INFINITE CLOSED MONOCHROMATIC SUBSETS OF A METRIC SPACE

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ABSTRACT. Given a coloring of the k-element subsets of an uncountable separable metric space, we show that there exists an infinite monochromatic subset which contains its limit point.

1. INTRODUCTION

Given a coloring of the k-element subsets of an infinite metric space X, the Infinite Ramsey Theorem guaranties the existence of an infinite monochromatic subset $\Lambda \subseteq X$. However, if $r \in X$ is a limit point of Λ, the Infinite Ramsey Theorem does not imply that $r \in \Lambda$. Therefore, it may be that Λ does not contain any of its limit points. We show that if additionally, X is assumed to be uncountable and separable, then such an infinite monochromatic set which contains its limit point exists.

The Topological Baumgartner-Hajnal Theorem, proved by Schipperus [3], provides a stronger result for the case $k = 2$, $X = \mathbb{R}$. The latter states that if the pairs of real numbers are colored with c colors, there is a monochromatic, well-ordered subset of arbitrarily large countable order type which is closed in the usual topology of $\mathbb{R}$. Applying the Topological Baumgartner-Hajnal Theorem to the special case where the order type is $\omega + 1$, provides Theorem 2.1 of this note for $k = 2, X = \mathbb{R}$. For $k = 3$ however, we show that the result in this note cannot be strengthened in some sense.

In Section 2 we state and prove Theorem 2.1. The proof relies on the Axiom of Choice. In Section 3 we show why the assumption that X is uncountable is required, and why we cannot expect a stronger result to hold: that there exists a monochromatic subset of X, which contains more than one of its limit points.

2. INFINITE CLOSED MONOCHROMATIC SUBSETS

Theorem 2.1. *Let X be an uncountable separable metric space with metric d. Let $k > 0, c > 0$, and let*

$$\chi : [X]^k \to \{1, 2, \ldots, c\}$$

be a coloring of the k-element subsets of X with c colors. Then there exists an infinite monochromatic subset $\Lambda \subseteq X$, and there exists $r \in \Lambda$ s.t. $\forall \epsilon > 0, \exists r_\epsilon \in \Lambda$ s.t. $d(r_\epsilon, r) < \epsilon$.

We first list the notation which is used in the proof:
- For a set A, $[A]^k$ denotes the set of k-element subsets of A.
- θ^s where $s \in \mathbb{N}, s \geq k - 1$ denotes a function
$$\theta^s : \{k - 1, k, k + 1, \ldots, s\} \to \mathbb{N}.$$
  $\mathcal{A}$ is the set of all such functions, for all $s > 0$.
- δ^s denotes a function which is defined on sets $\{i_1, i_2, \ldots, i_{k-1}\}$ where $i_1, \ldots, i_{k-1} \in \mathbb{N}$ and $i_1, \ldots, i_{k-1} \leq s$, and $\delta^s(\{i_1, i_2, \ldots, i_{k-1}\}) \in \{1, 2, \ldots, c\}$. Equivalently, δ^s is a coloring of the $(k - 1)$-element subsets of $\{1, \ldots, s\}$ with c colors. $\mathcal{B}$ is the set of all such functions, for all $s > 0$.
- Let j , $1 \leq j \leq c$, then
$$N_j(\{a_1, \ldots, a_{k-1}\}) := \{a \in X : \chi(\{a, a_1 \ldots, a_{k-1}\}) = j\}.$$
- For $x \in X, \varphi \in \mathbb{R}$, $B(x, \varphi)$ denotes the ball of radius φ around x with respect to d.
- $\{\Psi_n\}_{n \in \mathbb{N}}$: a countable subset of X which is dense in X. Such set exists by our assumption that X is separable.
- $\preccurlyeq$: well-ordering of X.

Proof overview.

(1) Suppose there exists a subset of X: $\{u_\alpha \in X : \alpha \in \mathbb{N} \cup \{\omega\}\}$, which satisfies the following properties:
  (a) Let $\alpha_1, \alpha_2, ..., \alpha_{k-1} \in \mathbb{N}$ be distinct numbers, then $\forall \alpha \in \mathbb{N}, \alpha > \alpha_{k-1}$.

  (2.1) $\chi(\{u_{\alpha_1}, \ldots, u_{\alpha_{k-1}}, u_\omega\}) = \chi(\{u_{\alpha_1}, \ldots, u_{\alpha_{k-1}}, u_\alpha\}).$

  (b) $\lim_{\alpha \to \infty} u_\alpha = u_\omega$.
     Then by a known construction of Erdös and Rado [4], it follows from property 1a that $\{u_\alpha \in X : \alpha \in \mathbb{N} \cup \{\omega\}\}$ contains an infinite monochromatic subset Λ such that $u_\omega \in \Lambda$.
     By setting $r = u_\omega$ it follows from property 1b that Λ satisfies the requirements of the theorem. Therefore it remains to show that there exists such set $\{u_\alpha \in X : \alpha \in \mathbb{N} \cup \{\omega\}\}$.
(2) Suppose on the contrary, that such set $\{u_\alpha \in X : \alpha \in \mathbb{N} \cup \{\omega\}\}$ does not exist.
(3) Define a function

(2.2) $f : X \backslash \{x_1, x_2, \ldots, x_{k-1}\} \to \mathcal{A} \times \mathcal{B},$

  where $x_1, x_2, \ldots, x_{k-1}$ are the first minimal elements in X according to $\preccurlyeq$. In other words, f assigns two finite functions every $x \in X$, except for the first $k - 1$ most minimal elements of X.
(4) Show that the function f is one-to-one.
(5) The above is contradiction, since by the theorem assumption X is uncountable, while $\mathcal{A} \times \mathcal{B}$ is countable.

Proof of Theorem 2.1.

Proof. Suppose that there exists a subset $\{u_\alpha \in X : \alpha \in \mathbb{N} \cup \{\omega\}\}$ which satisfies properties 1a and 1b as stated in step 1 of the proof overview. Then as described in step 1 of the proof overview, the theorem follows. Hence it remains to prove that such subset $\{u_\alpha \in X : \alpha \in \mathbb{N} \cup \{\omega\}\}$ exists.

For the sake of completeness, we now describe briefly the construction of Erdös and Rado which is mentioned in the proof overview. First, we define a coloring $\tilde{\chi}$, of the $(k-1)$-element subsets of $\{u_\alpha \in X : \alpha \in \mathbb{N}\}$ by $\tilde{\chi}(\{u_{\alpha_1}, \ldots, u_{\alpha_{k-1}}\}) = \chi(\{u_{\alpha_1}, \ldots, u_{\alpha_{k-1}}, u_\omega\})$. Now applying the Infinite Ramsey Theorem to this coloring, we conclude that there exists an infinite subset $\tilde{\Lambda}$ of $\{u_\alpha \in X : \alpha \in \mathbb{N}\}$, which is monochromatic with respect to $\tilde{\chi}$. Then $\tilde{\Lambda} \cup \{u_\omega\}$ forms an infinite subset of X which is monochromatic with respect to χ. Hence taking $\Lambda = \tilde{\Lambda} \cup \{u_\omega\}$ we get Λ as required.

As stated step 2 of the proof overview, we now assume that such set does not exist.

We now proceed by defining a function f as described in step 3 of the proof overview:

Let $x \in X$. We now define $f(x)$. We first define θ^s, δ^s, where $\theta^s \in \mathcal{A}$, $\delta^s \in \mathcal{B}$ for some $s > 0$, and $f(x)$ will be defined by $f(x) = (\theta^s, \delta^s)$.

For the purpose of defining θ^s, δ^s, we define recursively for each $i \le s+1$, a set $U_i \subseteq X$ and $u_i \in U_i$. The general idea is that we want to encode some information in θ^s, δ^s so that later we can reconstruct U_i and u_i without knowing x.

Let $u_1, u_2, \ldots, u_{k-1} \in X$ be the first $k-1$ minimal elements in X according to $\preccurlyeq$. Let $U_{k-1} = X$.

Now, suppose we already defined U_j, u_j for $1 \le j \le i$ and

$$\delta^s(\{j_1, j_2, \ldots, j_{k-1}\}), \theta^s(j)$$

for $k-1 \le j \le i-1$, $1 \le j_1, \ldots, j_{k-1} \le i-1$. We now define U_{i+1}, u_{i+1}, $\theta^s(i)$ and

$$\delta^s(\{i, j_1, \ldots, j_{k-2}\}), \forall j_1, \ldots, j_{k-2} \le i-1.$$

Let Ψ_n where n is the minimal number such that the following holds:

$$(2.3) \qquad\qquad x \in B(\Psi_n, 2^{-i})$$

Define

$$(2.4) \qquad\qquad \theta^s(i) = n.$$

Define

$$(2.5) \qquad \delta^s(\{i, j_1 \ldots, j_{k-2}\}) := \chi(\{x, u_i, u_{j_1}, \ldots, u_{j_{k-2}}\}),$$

and define

(2.6)
$$U_{i+1} := \bigcap_{j_1,\ldots,j_{k-2}<i} N_{\delta^s(\{i,j_1\ldots,j_{k-2}\})}(\{u_i, u_{j_1} \ldots, u_{j_{k-2}}\}) \cap B(\Psi_{\theta^s(i)}, 2^{-i}) \cap U_i$$

Define u_{i+1} to be the minimal element in U_{i+1} according to $\preccurlyeq$.

We have the following observations:

- $\forall i, x \in U_i$. In particular, U_i is not empty.
- After a finite number of steps $s > 0$, we must get to a situation where x is the minimal element in U_{s+1}. Otherwise, by setting $u_\omega = x$, we get an infinite sequence $\{u_\alpha \in X : \alpha \in \mathbb{N} \cup \{\omega\}\}$ which satisfies the properties as described in step 1 of the proof overview, which is a contradiction to our assumption.

after s steps, we define $f(x) = (\theta^s, \delta^s)$.

Step 4 in the proof overview: it remains to show that f is one-to-one. Suppose we are given $f(x)$, or in the notation above θ^s, δ^s. In order to show that f is one-to-one, we need to show that we can determine x. In the process described above, the same $U_1, \ldots, U_{s+1}$ and $u_1, \ldots, u_{s+1}$ can be recovered without knowing x, since θ^s, δ^s are known and by Eq. 2.6 for each $i \leq s$ we can calculate U_{i+1}, and u_{i+1} is the minimal element in U_{i+1}.

Therefore after reconstruction of the same s steps we get U_{s+1}, and $x = u_{s+1}$ is the minimal element of U_{s+1}. In other words, if x_1, x_2 are such that $f(x_1) = f(x_2)$, then both x_1 and x_2 must be the minimal element of U_{s+1}. Since U_{s+1} does not depend on x_1, x_2 we must have $x_1 = x_2$. As stated in the proof overview this is a contradiction.

$\square$

3. Counterexamples

The following example shows a coloring for the case $k = 3$, where any monochromatic subset has at most one limit point. Hence, for $k = 3$ we cannot always expect a monochromatic set which contains more than one limit point of itself. In this sense, Theorem 2.1 cannot be strengthened.

Example 3.1. Define a coloring of the 3-element subsets of $\mathbb{R}$ by

(3.1) $\chi(\{x_1, x_2, x_3\}) = \begin{cases} 0 & \text{if } |x_1 - x_2| \leq |x_2 - x_3| \\ 1 & \text{if } |x_1 - x_2| > |x_2 - x_3| \end{cases}$,

where $x_1 < x_2 < x_3$.

Suppose $\Lambda \subseteq \mathbb{R}$ is an infinite subset of $\mathbb{R}$ which has two distinct limit points: $l_1, l_2 \in \mathbb{R}$, where $l_1 < l_2$ and $|l_1 - l_2| = h$. We now show that Λ cannot be monochromatic. Let $r_1 \in \Lambda$ s.t. $|r_1 - l_l| < \frac{h}{5}$. Let $r_2, r_3 \in \Lambda$ s.t. $r_2 < r_3$ and $|r_2 - l_2| < \frac{h}{5}, |r_3 - l_2| < \frac{h}{5}$. Then $|r_1 - r_2| > h - \frac{h}{5} - \frac{h}{5} = \frac{3h}{5}$ while $|r_2 - r_3| \leq |r_3 - l_2| + |r_2 - l_2| \leq \frac{h}{5} + \frac{h}{5} = \frac{2h}{5}$. Hence $|r_1 - r_2| > |r_2 - r_3|$ and $\chi(\{r_1, r_2, r_3\}) = 1$. On the other hand, by a symmetric argument, taking $r_1, r_2 \in \mathbb{R}$ s.t. $r_1 < r_2, |r_1 - l_1| < \frac{h}{5}, |r_2 - l_1| < \frac{h}{5}$ and $r_3 \in \mathbb{R}$ s.t. $|r_3 - l_2| < \frac{h}{5}$,

we get $|r_1 - r_2| < |r_2 - r_3|$. Hence $\chi(\{r_1, r_2, r_3\} = 0$. This shows that Λ is not monochromatic.

In the following two examples we show why the assumption that X is uncountable is required in Theorem 2.1. This is done by defining a coloring of the k-element subsets of $\mathbb{Q}$, for which an infinite monochromatic subset does not contain any of its limit points. In the following two examples we assume $(\Psi_n)_{n \in \mathbb{N}}$ is an enumeration of $\mathbb{Q}$, and $\preccurlyeq$ is an order on $\mathbb{Q}$, defined by $\Psi_i \preccurlyeq \Psi_j$ if and only if $i \leq j$.

Example 3.2 is the construction of Sierpinsky [1], applied to $\mathbb{Q}$.

Example 3.2. Define a coloring of pairs of numbers in $\mathbb{Q}$ by

(3.2)
$$\chi(\{x_1, x_2\}) = \begin{cases} 0 & \text{if } x_2 \preccurlyeq x_1 \\ 1 & \text{if } x_1 \preccurlyeq x_2 \end{cases},$$

where $x_1 < x_2$.

Example 3.3. Define a coloring of 3-element subsets of $\mathbb{Q}$ by

(3.3)
$$\chi(\{x_1, x_2, x_3\}) = \begin{cases} 0 & \text{if } x_1 \preccurlyeq x_2 \text{ and } x_3 \preccurlyeq x_2 \\ 1 & otherwise \end{cases},$$

where $x_1 < x_2 < x_3$.

We claim the following.

Claim 3.4.   (1) In the coloring of Example 3.2 there exist no infinite monochromatic subset which contains its limit point.
  (2) In the coloring of Example 3.3 there exists no infinite subset, all whose 3-element subsets are colored 1, which contains its limit point, and there exists no subset, all whose 3-element subsets are colored 0 which contains more that 3 elements.

Part 1 of Claim 3.4 shows that for coloring 2-element subsets the requirement in Theorem 2.1 that X is uncountable, is necessary.

We may ask whether for some $l \geq k$ it holds that when coloring the k-element subsets of $\mathbb{Q}$ there must be either a subset of size l, all whose k-element subsets are colored 0, or an infinite subset which contains its limit point, all whose k-element subsets are colored 1. For $l = k$ this holds trivially. Part 2 of Claim 3.4 shows that for $k = 3$, this does not generally hold for $l > 3$.

We first prove the following claim, which we use in the proof of Claim 3.4.

Claim 3.5. If Λ is an infinite subset of $\mathbb{Q}$ which contains its limit point, then there exist $r_1, r_2, r_3 \in \Lambda$ s.t. $r_1 < r_2 < r_3$, $r_1 \preccurlyeq r_2$ and $r_3 \preccurlyeq r_2$.

Proof. Let r_3 be a limit point of Λ. For $\varphi \in \mathbb{R}$, Let $\Lambda'_\varphi = \{b \in \Lambda : \varphi < b < r_3\}$ and let $\Lambda''_\varphi = \{b \in \Lambda : \varphi > b > r_3\}$. Since r_3 is a limit point of Λ, it must be that either Λ'_φ is not empty for all $\varphi < r_3$, or Λ''_φ is not empty for all $\varphi > r_3$.

Assume first that Λ'_φ is not empty for all $\varphi < r_3$. There exist only finitely many $x \in \Lambda'$ s.t. $x \preccurlyeq r_3$. Hence for some $\varphi_1 \in \mathbb{R}$, $0 < \varphi_1 < r_3$, $\forall x \in \Lambda'_{\varphi_1}$, we have $r_3 \preccurlyeq x$. Let $r_1 \in \Lambda'_{\varphi_1}$. There exists φ_2 s.t. $r_1 < \varphi_2 < r_3$ and $\forall x \in \Lambda'_{\varphi_2}$, we have $r_1 \preccurlyeq x$. Let $r_2 \in \Lambda'_{\varphi_2}$. Then $r_1 < r_2 < r_3$ and $r_3 \preccurlyeq r_1 \preccurlyeq r_2$ as desired.

Now, if Λ''_φ is not empty for all $\varphi > r_3$, by a symmetric argument it follows that $r_3 < r_2 < r_1$ and $r_3 \preccurlyeq r_1 \preccurlyeq r_2$. Swapping r_1 and r_3 gives us that r_1, r_2, r_3 satisfy the requirements as desired. $\qquad\square$

We now show why Claim 3.4 holds.

Proof.   (1) Let χ be a coloring as in Example 3.2. Let Λ be a subset of $\mathbb{Q}$ which contains its limit point. Let $r_1, r_2, r_3 \in \Lambda$ as in Claim 3.5. Then $\chi(\{r_1, r_2\}) \neq \chi(\{r_2, r_3\})$. Hence the claim follows.

(2) Let χ be a coloring as in Example 3.3, and let Λ be an infinite subset of $\mathbb{Q}$ which contains its limit point. Let $r_1, r_2, r_3 \in \Lambda$ as in Claim 3.5. Then $\chi(\{r_1, r_2, r_3\}) = 0$. Hence not all 3-element subsets of Λ are colored 1. On the other hand, suppose on the contrary that there exist $r_1, r_2, r_3, r_4 \in \mathbb{Q}$ s.t. $r_1 < r_2 < r_3 < r_4$ and all 3-element subsets of $\{r_1, r_2, r_3, r_4\}$ are colored 0. Then considering r_1, r_2, r_3 we must have $r_3 \preccurlyeq r_2$. On the other hand, considering r_2, r_3, r_4 we have $r_2 \preccurlyeq r_3$. The last is a contradiction. Hence such monochromatic subset of size 4 does not exist.

$\qquad\square$

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